Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 30

Lecturer: Hunter Stufflebeam

Textbook Sections: None!

The Van Meegeren Art Forgeries, Radioactivity, and 1st Order Linear ODEs

(following M. Braun "Diff" Eqtns and Their Applications" 3rd Ed.)



story begins in 1889 with the birth of Henricus "Han" van Meegeren in Deventer, Netherlands. The son of a rural schoolmaster, van Meegeren initially traveled to Delft to study architecture under his father's strong suggestion. Van Meegeren, however, was more interested in studying art and making a name for himself in that way. So, without ever completing his architecture studies, in 1913, he moved his newly formed family to The Hague and began working as an artist, specializing in the techniques of the old Dutch masters like the celebrated Jan Vermeer.

Although talented, his career never took off as he had hoped, and he began to deal art as well. Then, in 1937, he approached a friend with a painting he had "found" tucked safely away for 100s of years in a castle — a lost Vermeer painting called "Jesus and His Disciples at Emmaus!" After a leading art historian an Vermeer authority, Abraham Bredius, both authenticated the painting and hailed it as Vermeer's masterpiece,

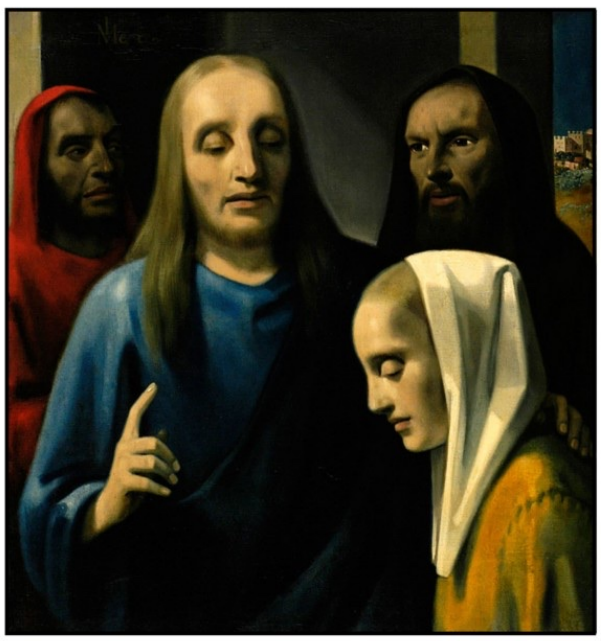
van Meegeren was able to sell his sensational find to a museum for about $280,000 at the time (several million today).

Of course, Vermeer never painted a work called "Disciples at Emmaus." This lost, finest work of Vermeer was nothing other than van Meegeren's brilliant forgery, and it had been a tremendous success.



"Christ and His Disciples at Emmaus"

Spurred on by this success, van Meegeren produced several more impeccable forgeries including a "Vermeer" titled "Christ and the Adulteress."



In 1943, van Meegeren sold it through a broker to the Nazi Field Marshall, and Hitler's right hand man, Hermann Göring. In return, van Meegeren received 137 looted Dutch paintings.

Then in 1945, Allied forces discovered Goering's stash of looted art in a salt mine, and amongst the ~7000 pieces was the remarkable "Vermeer." It was quickly traced back to van Meegeren, who was quickly arrested, charged with fraud, and aiding and abetting the enemy with threat of the death penalty.

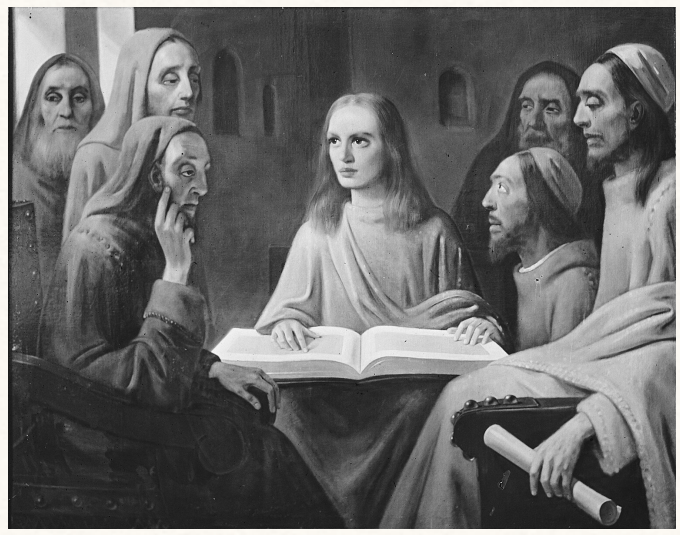
Unsurprisingly, van Meegeren eventually admitted to forging the

paintings, and positioned himself as a hero of Dutch culture for selling a fake painting to recover 137 legitimate Dutch works.

To help prove his claim, he painted a new forgery, "Jesus among the Doctors," in front of witnesses. In particular, he showcased

1) his use of authentic 17th cent. Dutch canvases as bases
2) his mixing of paints using traditional formulas from raw materials
3) his construction of badger hair paintbrushes, as used by Vermeer
4) his technique of mixing Bakelite into his paints so that when baked, the paint would harden just like very old oils
5) his technique of cracking and dying the canvas to age it appropriately

After investigating the forgeries, chemists detected traces of some modern pigments, and the Bakelite. So, van Meegeren's sentence was changed to forgery, he was sentenced to a year in prison, became a national hero, and died of a heart attack a year later.



"Jesus among the Doctors"

(By the way, van Meegeren was not really a great guy. He was actually pretty ok with the Nazis and their policies....)

Regardless of all of this, many art experts refused to accept the conclusion that "Disciples at Emmaus" was not a true Vermeer, because it was of such higher quality than the other forgeries. In 1967, Bernard Keisch and other chemists at CMU proved beyond a shadow of doubt that the work could not have been a Vermeer.

Radiometric/Radioactive Dating:

Goal: Use the phenomenon of radioactive decay to estimate the age of materials since their time of formation.

Key Idea: As discovered by Henri Becquerel and Marie and Pierre Curie in 1896, certain nuclides are unstable, and will decay into different nuclides probabilistically (a discovery for which they were awarded the 1903 Nobel Prize).

This process of decay releases energy in the form of radiation, which we can measure.

Since the rate of this decay is a property of the nuclide in question, and specifically of each atom of the nuclide in question, a natural basic model for this process is as follows (first written down by Rutherford-Soddy in 1902, who would also go on to win the Nobel)

- Let $N(t)$ = # atoms of the nuclide at time t

  ($N(t)$ is considered to be so large that, even though it is a discrete natural number, $t \mapsto N(t)$ is modeled as a smooth function).
- The decay of a given nucleus is random in time and memoryless (so that it is equally likely to decay at any given point in time). Thus, the number of disintegrations per unit time, $N'(t)$, should be proportional to the number of nuclei at time t, $N(t)$.
- As different nuclides have different rates of decay, a given nuclide carries a specific decay constant $\lambda > 0$. Thus, we can finally write our ODE model for the changing number of nuclides in a population as it undergoes radioactive decay:

$$\frac{dN}{dt} = -\lambda N$$

- $\frac{dN}{dt}$: rate of change of # of atoms
- λ: decay constant of nuclide
- N: current number of atoms
- negative sign signifies that the number of atoms decreases.

Let's Solve this ODE problem:

Let's suppose we start, at time t_0, with $N_0 = N(t_0)$ atoms of some nuclide. This gives us the initial value problem

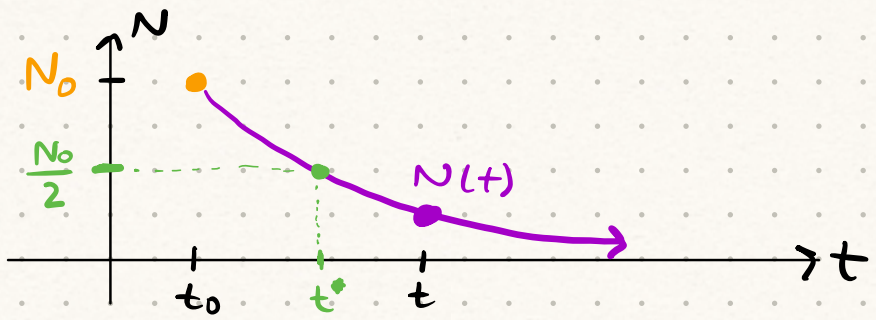
$$(\star)\quad \begin{cases} N'(t) = -\lambda N & \text{for } t \in [t_0, \infty) \\ N(t_0) = N_0 \end{cases}$$

To solve this, we divide by N and integrate in t, using the initial condition $N(t_0)=N_0$ to solve for the arbitrary constant of integration:

$$\frac{N'(t)}{N(t)} \overset{(\star)}{=} -\lambda \Rightarrow -\lambda(t-t_0) = \int_{t_0}^{t} \frac{N'(s)}{N(s)}\,ds = \log N(t)\Big|_{t_0}^{t} = \log \frac{N(t)}{N_0}$$

or in other words:

$$\begin{cases} N(t) = N_0 \cdot e^{-\lambda(t-t_0)} \\ \text{for} \quad t \geq t_0. \end{cases}$$



Remark/Definition: There is a particularly useful way to characterize how quickly a nuclide decays, called its **half life**. This is the time $t^* - t_0$ it takes for **half** of the original atoms to decay:

$$\frac{N_0}{2} = N(t^*) = N_0 e^{-\lambda(t^*-t_0)}$$

In other words, $\text{half life} = (t^* - t_0) = -\dfrac{\log(\frac{1}{2})}{\lambda} = \dfrac{1}{\lambda}\log(2)$.

The upshot: We can compute the decay constants/half lives of many nuclides of interest. As is relevant for our analysis of paintings, consider the following abbreviated decay series for uranium-238 (which naturally occurs throughout the Earth's crust and in particular feeds the supply of radioactive radium-226 and lead-210 naturally found in the pigment white lead used by artists for >2000 years)

$$^{238}U \xrightarrow{\sim 4.5 \text{ billion yrs}} \cdots \rightarrow {}^{226}Ra \xrightarrow{\sim 1600 \text{ yrs}} \cdots \rightarrow \underbrace{{}^{218}Po}_{\text{(important for later)}} \xrightarrow{\text{this class}} \cdots \rightarrow {}^{210}Pb \xrightarrow{\sim 22 \text{ yrs}} \cdots \rightarrow {}^{206}Pb$$

Stable

ie. it takes ~4.5 bil. yrs for half the initial amount of ^{238}U to decay into ^{226}Ra, and it takes ~1600 yrs for half of a given amount of ^{226}Ra to decay into ^{210}Pb, and finally ~22 yrs for half a given amount of the ^{210}Pb to decay into the stable nuclide ^{206}Pb.

In particular, as ^{210}Pb decays, it is replenished by the decay of ^{226}Ra, which is in turn replenished by the decay of ^{238}U.

What is interesting for our application here: The manufacture of white lead pigment (by smelting lead ore) removes ~90% of all the ^{226}Ra and its descendants in the pigment, but preserves the ^{210}Pb. So, the newly manufactured pigment starts life with its steady supply of "fresh" ^{210}Pb cut off, while the starting amount of ^{210}Pb decays all the way to ^{206}Pb with a short half life of ~22 yrs.

If the pigment is old enough, the remaining amount of ^{226}Ra would steadily feed the supply of ^{210}Pb once again with the two populations in a new equilibrium ratio.

Let's model this scenario with an ODE

Let
- $N(t)$ = amount of ^{210}Pb nuclides per gram of white lead pigment at time t years.
- $N_0 = N(t_0)$ = amount of ^{210}Pb nuclides per gram of white lead pigment at the time of manufacture t_0.
- $R(t)$ = (# of disintegrations of ^{226}Ra)/yr per gram of white lead pigment at time t years.

⚠ on a timescale of ~300 years (age of a true Vermeer), not much ^{226}Ra will have decayed (half life ~1600 yrs), so $R(t) \approx R$ constant for the time scale we care about in our model.

- λ = decay constant of ^{210}Pb. Since ^{210}Pb has a half life of ~22 yrs, we have that

$$22 \text{ yrs} = \frac{\log 2}{\lambda} \iff \lambda = \frac{\log 2}{22} \text{ yrs}^{-1}$$

Putting it all together:

$$(**) \quad N'(t) = -\lambda N(t) + R \quad \text{for} \quad t_0 \le t \quad \text{with} \quad N(t_0) = N_0$$

- $N'(t)$: total net rate of change of ^{210}Pb nuclides
- $-\lambda N(t)$: − rate of change of ^{210}Pb nuclides due to their decay
- $+R$: + rate of new ^{210}Pb nuclides from decay of ^{226}Ra
- $t_0 \le t$: domain of time in years since manufacture
- $N(t_0) = N_0$: initial condition = initial amt of ^{210}Pb

To solve this initial value problem:

We use a technique (which we'll introduce properly next week) called the integrating factor method:

- Let $\mu(t) = e^{\lambda t}$. Then $\mu'(t) = \lambda e^{\lambda t} = \lambda \mu(t)$.
- Magic happens (the product rule):

$$\begin{aligned} \frac{d}{dt}\left(N(t)\mu(t)\right) &= N'(t)\mu(t) + \lambda N(t)\mu(t) \\ &= \mu(t)\left[N'(t) + \lambda N(t)\right] \\ &\overset{(**)}{=} \mu(t) \cdot R \end{aligned}$$

- Hey, this new ODE is easy to solve! Just integrate:

$$\begin{aligned} N(t)\mu(t) - N(t_0)\mu(t_0) &= \int_{t_0}^{t} \frac{d}{ds}\left(N(s)\mu(s)\right) ds \\ &= \int_{t_0}^{t} \mu(s) \cdot R \, ds \\ &= \frac{R}{\lambda}\left(e^{\lambda t} - e^{\lambda t_0}\right) \end{aligned}$$

After algebra,

$$N(t) = N_0 \cdot e^{-\lambda(t-t_0)} + \frac{R}{\lambda}\left[1 - e^{-\lambda(t-t_0)}\right]$$

- $N(t)$: # ^{210}Pb nuclides at time t years
- $N_0 \cdot e^{-\lambda(t-t_0)}$: what we got in the original model for radioactive decay
- $\frac{R}{\lambda}\left[1 - e^{-\lambda(t-t_0)}\right]$: new term which "replenishes the supply of ^{210}Pb" when the ^{226}Ra decays.

As for what we can actually measure here:

- $N(t)$ via eg spectroscopy, etc.
- $R \approx R(t)$ via eg. a Geiger counter

What we can't directly measure:

- N_0 (we weren't there at the time of manufacture!)
- $t - t_0$ = age of the substance (what we want to know! it's either ~20 yrs or ~300 yrs)

So, lets do some algebra to isolate N_0:

$$\lambda N_0 = \lambda N(t) \cdot e^{\lambda(t-t_0)} - R\left(e^{\lambda(t-t_0)} - 1\right)$$

- λN_0: # of disintegrations of ^{210}Pb nuclei/yr @ time t_0 / (g of pigment)
- $\lambda N(t)$: # of disintegrations of ^{210}Pb nuclei/yr @ time t / (g of pigment)

In a nutshell, what happens next:

(0) If the pigment is old: The (# disints/yr)/(g of pigment) will be the same for both ^{226}Ra and ^{210}Pb. This is called secular equilibrium, and is due to the fact that $\lambda_{Ra} << \lambda_{Pb}$. For ^{226}Ra and ^{210}Pb, this equilibrium state is reached in ~150 years.

If the pigment is new: The tiny amount of ^{226}Ra in the pigment is outshined by the large amount of ^{210}Pb decaying at its rate of $\lambda N(t)$ (much higher than what it should be to be in equilibrium with the smaller proportion of ^{226}Ra)

(1) Lets suppose the painting was 300 years old, and plug in $t - t_0 = 300$ yrs to reflect that (also, plug in for $\lambda = \frac{\log(2)}{22}$)

$$(\lambda N_0) = (\lambda N(t)) \cdot e^{300\lambda} - R\left(e^{300\lambda} - 1\right)$$

$$\approx (\lambda N(t))\, 2^{\frac{150}{11}} - R\left(2^{\frac{150}{11}} - 1\right)$$

(2) Empirically, we can calculate R from samples of the painting, and find that

$$R(t) \approx R = (420{,}482 \text{ disints of } {}^{226}Ra/yr)/(g \text{ of pigment})$$

(3) Putting that together,

$$(\lambda N_0) \approx (\lambda N(t))\, 2^{\frac{150}{11}} = (420{,}482)(2^{\frac{150}{11}} - 1)$$

(4) To estimate $\lambda N(t)$, it ends up being easier to measure

$$(\#\text{ disints of } {}^{210}Po/yr)/(g \text{ of pigment})$$

(since the half life of ^{210}Pb is ~22 yrs and the half life of ^{210}Po is ~138 days, these guys reach secular equilibrium in a few years) since ^{210}Po's decay rate is easier to measure in a laboratory.

For "Disciples at Emmaus",

$$\lambda N(t) \approx (\#\text{ disints of } {}^{210}Po/yr)/(g \text{ of pigment})$$
$$\approx (4{,}467{,}600 \text{ disint's}(yr)/(g \text{ of pigment}).$$

(5) If we put this in the model and divide by 525,600 to convert years to minutes, we get

$$\lambda N_0 \approx 98{,}500 \ (\text{disints of } {}^{210}Pb/\text{min})/(g \text{ of pigment})$$

λN_0: rate of decay of ^{210}Pb at the birth of the pigment

(6) If that were the rate of decay of the population of ^{210}Pb nuclides at the formation of the pigment (where the amount of ^{210}Pb would have been in secular equilibrium with the amount of ^{238}U in the original lead ore), the concentration of ^{238}U in the original ore would be ~13.2% by mass, which is geologically absurd.

Thus, the pigment in "Disciples at Emmaus" couldn't be 300 years old!